Lebesgue Measure

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1 Motivation

I'll take a different approach than Wikipedia here, and start backwards by starting with integration, why we'd want to improve it, and then going back to improve the definitions to motivate and hopefully give a picture for outer measure and measure.

All the central motivations I've heard for Lebesgue measure involve improving Riemann integration. Namely, with a Lebesgue integral, you can do the following:

- Give an answer to $\int_{[0,1]} 1_Q$ or $\int_{[0,1]} C(x)$ (the integral of the identity function of the rationals or the Cantor function over the interval between 0 to 1).

- Properly define and extend the number of cases for when one can say $\int \lim = \lim \int$ (I think this is one of those 'abuse of notations' things, because even when all you know is the Riemann integral, you'll do things like this, but it's not really mentioned that what you're doing is not rigorous–sort of like how in differential equations you'll say $\frac{dx}{dt} = x$, therefore we 'multiply both sides by $\frac{dt}{x}$' to get $\frac{dx}{x} = dt$ even though that's not mathematically rigorous).

- You can generalize the notion of integration. 'Lebesgue measure' typically applies to the real number line, but you could have a 'counting measure' (which is exactly what it sounds like, the mesaure of a set is the number of elements in that set), in which case integration becomes a simple summation (which sounds dumb, but a lot of the results on different norms can get passed over to integrals using this, which is why Cauchy's inequality means so many different thing). More importantly to what I'm interested in, you can also define something similar to a Lebesgue measure over an event space (in which case 'integration' is the same thing as 'expectation value').

2 Picture

Riemannian integration is defined by breaking up the x-axis that you're integrating over, multiplied by the height at each of those intervals, and then sums that up–all as you shrink the size of the intervals you're limiting over.

Lebesgue integration does the opposite. It breaks up the range of the function into intervals, and then multiplies the average height of the interval in the range with the entire 'size' of the inverse of the function over the domain.

In short, Riemann integration is a sum of the product of images of points of a function by the sizes of the intervals over a partition of the domain, while Lebesgue integration is a sum of the product of the points of the range by the sizes of the preimages of the function of a partition of the range.

The whole reason I'm writing it this way is because the main picture I have in my head while thinking about Lebesgue integration and thereafter Lebesgue measure is this:

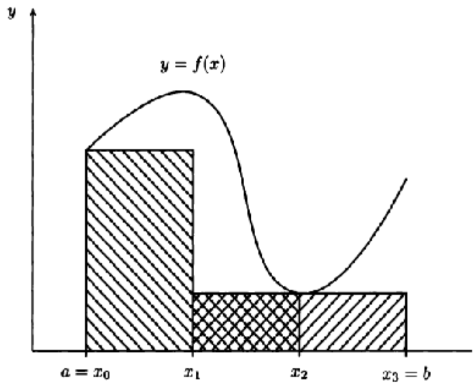

Fig. 1.3.1. Lower Riemann sum.

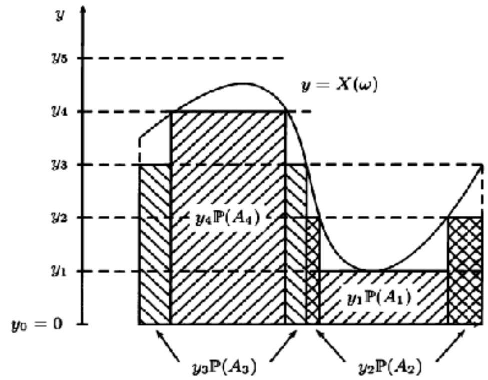

Fig. 1.3.2. Lower Lebesgue sum.

The reason and advantage is that doing it this way allows integration to be more function dependent. Before, we had to be worried about whether the function would be 'too discontinuous' (any function that has countable discontinuities, like the identity function of the rationals or the Weierstrass pay function, or the Cantor function, are not Riemann integrable), but now we don't have to be, because we're defining the terms of the integration dependent on the function. For a similar reason, for certain limits of functions, we can define their limit of their integrals as well. This is why we get a lot of the things we want as a consequence mentioned in the previous section.

In order to define integration as a sum of the sizes of the pre-images of a function, we need to find the most arbitrary definition of the size of a set on the real number line.

There are two aspects we need to get the picture from the previous section to be well-defined.

First is a more general notion of the size of a set. When we were doing Riemannian integration, the partition of a set always consisted of a finite number of intervals taken under a limit, and the size of an interval is rather straightforward to define. Under Lebesgue integration, we may have an incredibly strange preimage of a function (think Weierstrass pay function).

The second problem is that we still want basic integration consistency. I want to be sure that
$\int_{[0,2]} f d\mu$
isn't suddenly going to be different than
$\int_{[0,1]} f d\mu + \int_{[1,2]} f d\mu$
Ensuring the first is where covering and outer measures come into play.
Ensuring the second is where measures and Lebesgue spaces come into play.

3 Outer Measures

3.1 Covering

A priori, the preimage of a given function may be incredibly chaotic. On the other hand, we are motivated by the example of Riemann integration, and that the size of the interval (a, b) is $b - a$. We'd like to be able to keep our notion of the size of a simple interval while giving us some sort of tool to deal with more complicated sets.

A covering of a set A is an open set which is a countable collection of intervals $I_j = (a_j, b_j)$–not necessarily disjoint–such that $\cup_j I_j \supseteq A$.

Because $\cup_j (-j, j) = \Re \supseteq A$, every set has a covering.

For more general spaces, this usually implies that measure spaces naturally identify with the underlying topology, which gets into metrics/etc., but I won't get into that here.

3.2 Outer Measure

Given a covering I_j of a set A it is natural to define the outer measure to be $\mu*(A) := \Sigma_j |b_j - a_j|$, but since coverings are not necessarily unique, we instead say that:

$\mu*(A) := inf_{\cup_j I_j \supseteq A} \Sigma_j |b_j - a_j|$

In general, the way we compute an outer measure of something is first we create a sequence of coverings for it, and then use those coverings to calculate the outer measure.

3.3 Examples

Consider the rational numbers Q. Because it is countable, let r_j be the sequence of all rational numbers. Then $(r_j - \frac{1}{2^{j+k}}, r_j + \frac{1}{2^{j+k}})$ covers Q, thus

$\mu*(Q) \leq inf_k \Sigma_{j=0}^{\infty} 2^{-j-k+1} = inf_k 2^{2-k} = 0$

Consider the Cantor set C, where each middle third of the set $[0, 1]$ is progressively removed. E.g., $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup \frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and so on. Then C_1 is coverable by 2^1 sets of size 3^{-1}, C_2 is coverable by 2^2 sets of size 3^{-2}, and so on. Then by induction, C is coverable by a set whose outer measure is $(\frac{2}{3})^k$ for any given k, and therefore $\mu*(C) = 0$. This example is interesting because it means you can have an uncountable set with measure 0.

4 Measures

4.1 Adding

The central problem with outer measures is that $\mu*(\cup_i A_i) \leq \Sigma_i \mu*(A_i)$ and not necessarily equality even with disjoint A_i. Measures come about solely because we want to consider the space of sets for which $\mu*(\cup_i A_i) = \Sigma_i \mu*(A_i)$ for disjoint A_i. An outer measure under such a space is consider a measure.

I am exposing this material a bit differently. Most books define measure, and then say it has this additive property. I think it might be more intuitive to say that the space is limited so that measure is well-defined in the same way that 0 is exluded from the binary operation of division to make it well-defined.

In other words, outer measures and measures are the same thing, they just have different domains of operation. Therefore, is might be better to describe the Lebesgue nonmeasurable set example in the following way (this is probably the third slightly different description of a Lebesgue nonmeasurable set I've been through now in trying to find the simplest way to describe this).

4.2 Lebesgue Nonmeasurability

The following is an example of a collection of disjoint sets for which $\mu*(\cup_i A_i) < \Sigma_i \mu*(A_i)$. Assume it was always equal.

Let $\alpha \in \Re \setminus Q$ and $P_\alpha := \alpha + Q$ modulo 1. By axiom of choice, one can construct a disjoint set of such P_α that partition $[0, 1]$. And by axiom of choice, choose one element from each P_α and call this set P.

Note that by definition that $P + Q$ modulo $1 = [0, 1]$, or if r_i is an enumeration of Q that $\cup_i(P + r_i) = [0, 1]$.

Because outer measure is invariant under translation (the translation of the covering is still a valid covering), then $1 = \mu*([0, 1]) = \mu*(\cup_i(P + r_i)) = \Sigma_i \mu*(P + r_i) = \Sigma_i \mu*(P) = \Sigma_i C$, which is either 0 (if $C = 0$) or unbounded (if $C \neq 0$), not 1.